Introduction to Mathematical Quantum Theory

Text of the Exercises

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Exercise 1

Let $\mathcal{H}$ be an Hilbert space. Let A and B linear operators on $\mathcal{H}$ such that there exists $\alpha \in \mathbb{C}\setminus\{0\}$ such that

$$[A,B] = \alpha \, \mathrm{id}\,. \tag{1}$$

Prove that A and B cannot be both bounded.

Hint: Assume both bounded; consider $\|[A,B^n]\|$ and find an absurd.

Exercise 2

a Prove that for any $\alpha \in \mathbb{C}$ such that $\mathrm{Re}\,(\alpha) > 0$,

$$\left(\int_{\mathbb{R}} e^{-\frac{x^2}{2\alpha}} dx\right)^2 = \int_{\mathbb{R}^2} e^{-\frac{x^2+y^2}{2\alpha}} dxdy \tag{2}$$

$$= 2\pi\alpha, \tag{3}$$

where the integral over $\mathbb{R}^2$ can be evaluated using polar coordinates. Deduce that

$$\int_{\mathbb{R}} e^{-\frac{x^2}{2\alpha}} dx = \sqrt{2\pi\alpha}, \tag{4}$$

where the square root is the one with positive real part.

b For all $B \geqslant A > 0$ and $\alpha \in \mathbb{C}\setminus\{0\}$ we have

$$\int_A^B e^{-\frac{x^2}{2\alpha}} dx = -\left.\frac{\alpha}{x} e^{-\frac{x^2}{2\alpha}}\right|_A^B - \int_A^B \frac{\alpha}{x^2} e^{-\frac{x^2}{2\alpha}} dx. \tag{5}$$

Using this, prove that the integral in (4) is convergent for all nonzero α with $\mathrm{Re}\,(\alpha) \geqslant 0$, provided the integral is interpreted as a principle value when not absolutely convergent, where the principal value is defined as

$$\mathrm{PV} \int_{\mathbb{R}} f(x)\, dx := \lim_{R\to\infty} \int_{-R}^{R} f(x)\, dx. \tag{6}$$

c Prove that the result of **a** is also valid for nonzero values of α with $\mathrm{Re}\,(\alpha) = 0$, at least in the principal value.

Hint: Given $\eta \neq 0$, show that the principal value from A to $+\infty$ of $\exp\left[-\frac{x^2}{2(\gamma+i\eta)}\right]$ is small for large A, uniformly in $\gamma \in [0,1]$.

d Prove that

$$\frac{1}{2\pi} \mathrm{PV} \int_{\mathbb{R}} e^{ikx} e^{-i\frac{\hbar t}{2m}k^2} dk = \sqrt{\frac{m}{2\pi i \hbar t}} e^{i\frac{m}{2\hbar t}x^2}, \tag{7}$$

where the square root is the one with real positive part.

Exercise 3

Consider a separable Hilbert space $\mathcal{H}$ and a complete orthonormal system for it $\{\varphi_n\}_{n\in\mathbb{N}}$. Assume that φ_∞ cannot be written as a finite linear combination of elements of $\{\varphi_n\}_{n\in\mathbb{N}}$. Let D denote the dense linear subspace of $\mathcal{H}$ consisting of all finite linear combinations of elements of $\{\varphi_n\}_{n\in\mathbb{N}}$ and of φ_∞. On D define the operator $T : D \to \mathcal{H}$ defined as

$$T\left(\alpha_\infty\varphi_\infty + \sum_{n\in\mathbb{N}} \alpha_n\varphi_n\right) := \alpha_\infty\varphi_\infty. \tag{8}$$

Prove that T is not bounded.

Hint: Use the closed graph theorem.

Exercise 4

Recall the definition of $H^2(\mathbb{R})$ as

$$H^2(\mathbb{R}) := \left\{\psi \in L^2(\mathbb{R}) \mid k^2\widehat{\psi} \in L^2(\mathbb{R})\right\}$$

Recall that in class we defined the map that to any initial datum $\psi_0 \in L^2(\mathbb{R})$ would associate $\psi_t := \widetilde{U}_0(t)\psi_0$, defined via the Hamiltonian $H_0 := -\frac{\partial^2}{\partial x^2}$ with domain $\mathcal{D}(H_0) = H^2(\mathbb{R})$. Indeed if $U_0(t)\psi_0$ is defined for any $\psi_0 \in \mathcal{S}(\mathbb{R})$ as the unique solution to

$$\begin{cases} i\hbar\partial_t\left(U_0(t)\psi_0\right) = H_0U_0(t)\psi_0 \\ U_0(t)\psi_0\big|_{t=0} = \psi_0, \end{cases} \tag{9}$$

then $\widetilde{U}_0(t)$ is defined by density on the whole space $L^2(\mathbb{R})$, and coincides with $U_0(t)$ on $\mathcal{S}(\mathbb{R})$.

Prove that if $\psi_0 \in D(H_0)$ then $\psi_t \in \mathcal{D}(H_0)$.